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The mathematical structure of theories of semi-geostrophic type

I. Roulstone and M.J. Sewell*

Numerical Weather Prediction, Meteorological Office,
London Road, Bracknell,
Berkshire RG12 2SZ. UK.

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Abstract

In the past twenty years, semi-geostrophic equations have become a prominent model for describing certain atmospheric motions on a synoptic scale, including the presence of fronts. Theoretical studies of them have revealed Hamiltonian features, and numerical solutions motivated by the need to improve weather forecasts have been explored. A shallow water theory analogue has been used as a paradigm for some aspects.

This paper sets out to uncover the essence of the mathematical structure of semi-geostrophic theory. We study the shallow water and atmospheric theories side-by-side, and we introduce a generalized form which encapsulates the differences between them. When the Coriolis parameter f is a constant, it is found that a lift transformation is at the heart of the theory, and the consequences of this are developed. When f is not a constant, the role of the lift transformation is, in some respects, looser; we explore the extent to which it still offers a worthwhile guide. In particular, it can be viewed as motivating a generalization of the geostrophic momentum transformation for planetary semi-geostrophic equations.

The paper is broadly self-contained, and it takes account of several different strands in the existing literature.

1 Introduction

The semi-geostrophic equations are an approximation to the exact momentum balance equations of a rotating fluid, in which the acceleration is replaced by the time derivative (following the particle) of the so-called geostrophic velocity. They

*Permanent address: Department of Mathematics, University of Reading, Whiteknights, P.O. Box 220, Reading RG6 6AF.

are regarded as a good approximation for certain three dimensional atmospheric motions on a synoptic scale, where the hydrostatic approximation is used in the vertical direction, and including motions in which discontinuities modelling fronts can occur; and for certain two dimensional shallow water motions over a rotating bed.

Physical arguments, and in some respects rather detailed ones, are naturally required to justify this "geostrophic momentum" approximation, as it was called by Hoskins (1975, §3) in his account of them for the atmospheric case. He seems to have been the first to use the adjective "semi-geostrophic" to describe the resulting equations (contained in his (10)) for constant Coriolis parameter, and he remarked that they had also been introduced by Eliassen (1948) and Fjørtoft (1962).

Hoskins (*op. cit.*, §4) and (in one dimension) Eliassen (1962) also introduced a change of independent variables, from cartesian coordinates of the particle to "geostrophic coordinates". Hoskins showed that their time derivatives following the particle are the geostrophic velocity components. It was the system of equations expressed, after use of the geostrophic momentum approximation, in terms of these new variables that Hoskins originally called semi-geostrophic. Conservation of energy applies. For adiabatic flow, potential temperature is conserved. When the Boussinesq approximation is also made, potential vorticity is conserved too.

In the course of constructing solution strategies for such equations, the work of Cullen and co-workers (e.g. Cullen and Purser (1984), Purser and Cullen (1987), Shutts and Cullen (1987)) began to reveal features reminiscent of Hamiltonian mechanics, such as energy minimization techniques and duality, in addition to conservation properties. Hamilton's equations themselves are not stated, and the connexion with the first derivation of the semi-geostrophic equations for shallow water theory from Hamilton's principle by Salmon (1983, 1985) remained obscure. In the atmospheric context the connexion with Hamilton's equations was viewed at first more as folklore rather than fact. Chynoweth, Porter and Sewell (1988), and Chynoweth and Sewell (1989, 1991), made explicit the connexion of the semi-geostrophic equations with the Legendre dual transformation and Hamilton's equations. Purser (1988) related these approaches to that of Salmon (*op. cit.*). The Hamiltonian is an unknown function of the geostrophic coordinates. It needs to be found by solving a Monge-Ampère problem, in which the potential vorticity reciprocal is viewed as an assigned function of the geostrophic coordinates at each time, in the manner reviewed by Cullen, Norbury and Purser (1991).

Sewell and Roulstone (1993, 1994) began a study of certain basic aspects of transformation theory in Hamiltonian dynamics, namely canonical, contact and lift transformations. This was with the long-term object of contributing to an understanding of the semi-geostrophic equations, having noted a remark by Blumen (1981) about a contact transformation property of them. It emerged that *the geostrophic coordinate transformation is part of a lift transformation.* (The contact property is secondary).

In this paper we use that fact as a basis for a study of the mathematical structure of semi-geostrophic equations in several contexts. The object is to complement the physical reasons for their adoption with the mathematical properties which flow from the hypothesis of a lift transformation as a starting point. We introduce concrete and abstract notations to run side-by-side, as a way of deliberately detaching the mathematics from the physics, and then bringing them together again. In particular, this allows us to define and work with generalized semi-geostrophic equations. The method conveys a very clear indication of how generic features of f-plane semi-geostrophic theory follow as a consequence of the lift transformation. It also allows us to go some way towards an understanding of what to expect about the structure of non-f-plane theories. Among other results, we show how the planetary semi-geostrophic equations of Shutts (1989) follow from a non-f-plane generalization of the geostrophic coordinate transformation. Non-f-plane shallow water theories were studied by Roulstone and Sewell (1996) using a different approach.

There is a sense in which the approach *via* lift transformations proposed here starts at the opposite end from that represented by hypothesizing a Hamilton principle, and exploring its consequences, which is a more traditional method. Of course, ours is not the only approach which may be informative, but it has not been exploited before.

We include brief summaries of the shallow water and atmospheric theories, so that the paper is reasonably self-contained.

2 Thermodynamics and constitutive properties

In this Section we describe properties which a representative particle is deemed to have in any motion.

There are two distinct contexts which we have particularly in mind from the outset, in both of which the semi-geostrophic approximation is often applied. The definition of a material particle is different in the two cases.

One context is the mid-latitude motion of the atmosphere on a synoptic scale, which includes phenomena of direct interest to the meteorologist. The other context is the motion of shallow water over a rotating bed, which is often regarded as a useful mathematical paradigm of the atmospheric problem, but which lacks the ability to describe certain vertical effects, and which also replaces incompressibility by compressibility.

2.1 Atmospheric theory

In this context the particle is a parcel of air which is free to move in three dimensions. A cartesian reference frame fixed to the Earth is used as the standpoint from which to describe particle motions. At time t the particle has height h above a local tangent plane to the Earth's surface, and horizontal distances x and y (for example, east and north) measured parallel to that plane. Particle labels a, b and c are defined as the values of x, y and h at some particular time, say $t = 0$, and they identify the particle at all subsequent times.

A motion of the atmosphere is defined by three equations

$$x = x(a,b,c,t),\ y = y(a,b,c,t),\ h = h(a,b,c,t) \tag{1}$$

relating the values of x, y and h on the left to functions, usually unknown in advance, of a, b, c, and t on the right.

Standard thermodynamics endows every particle, for all time and for every motion, with the same function $H(\eta, p)$ representing enthalpy per unit mass, such that

$$\tau = \frac{\partial H}{\partial \eta},\ \frac{1}{\rho} = \frac{\partial H}{\partial p} \tag{2}$$

where η, p, τ and ρ are entropy, pressure, absolute temperature and density respectively. Thus the functions $H(\eta,p), \tau(\eta,p)$ and $\rho(\eta,p)$ do not depend explicitly on t or on a, b, c as well, so that their values can only vary when η or p do so.

For dry air the ideal gas model is usually assumed, in which $p = k\rho^\gamma$ emerges from $(2)_2$ when

$$H(\eta,p) = \sigma(\eta)\zeta(p) \tag{3}$$

is the particular product function in which

$$\zeta(p) = (\gamma/(\gamma-1))p^{(\gamma-1)/\gamma},\ \sigma(\eta) = k^{1/\gamma},\ k(\eta) = R\exp(\eta - d)/c_v,$$

where γ, R, d, c_v and c_p are given constants related by $R = c_p - c_v$ and $\gamma = c_p/c_v$.

For many years, e.g. since the work of Bjerknes in 1910, the pressure, or a function of it, has been a more convenient vertical coordinate for some purposes than the true particle height h. To define such a pseudo-height here, we first select arbitrary reference values η_0 and p_0 (say) of η and p, which imply reference values $H(\eta_0, p_0) = H_0$ (say) and $\rho(\eta_0, p_0) = \rho_0$ (say) from (2). Let g denote the combined acceleration due to gravity and the vertical effect of the Earth's rotation. Let z defined by

$$z = \left(1 - \frac{H(\eta,p)}{H(\eta,p_0)}\right)\frac{H_0}{g} = \left(1 - \left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}\right)\frac{\gamma p_0}{(\gamma-1)\rho_0 g} \tag{4}$$

denote the pseudo-height, in terms of a general enthalpy function in the first expression (Chynoweth and Sewell, 1991), and of the ideal gas in the second (Hoskins,

1971).

In terms of z we can interpret p_0 as the pressure at the bottom of the atmosphere, because $z = 0$ when $p = p_0$. If $H(\eta, 0) = 0$ as in the ideal gas, the pseudo-height of the atmosphere is H_0/g or $\gamma p_0/(\gamma - 1)\rho_0 g$, because z has that value when $p = 0$.

The height of the atmosphere is small compared with the radius of the Earth, so that, notwithstanding the possibility of gravity anomalies, we adopt the approximation that g is an absolute (positive) constant.

Then during any motion (1) with (3), z can only vary when p does so. In general, p will be a function $p(x, y, h, t)$ which is unknown in advance. Vertical acceleration is neglected, so that vertical momentum balance is achieved by satisfying the hydrostatic equation

$$\frac{\partial p}{\partial h} + \rho g = 0. \tag{5}$$

This ensures that the function $z(x, y, h, t)$ implied by $(4)_2$ has the property

$$\frac{\partial z}{\partial h} = \frac{\sigma_0}{\sigma} > 0 \tag{6}$$

whenever (3) applies with $\sigma(\eta) > 0$, where $\sigma_0 = \sigma(\eta_0)$, in particular for the ideal gas.

Therefore the inverse function theorem always permits the true height h to be expressed as a function $h(x, y, z, t)$, and the pseudo-height z can subsequently be used as an alternative vertical coordinate. The geopotential $gh = \phi$ (say) per unit mass can then be expressed as a function $\phi(x, y, z, t)$ such that

$$\frac{\partial \phi}{\partial z} = \frac{g\sigma}{\sigma_0}. \tag{7}$$

This is an alternative statement of vertical momentum balance, in the sense that (5) and (7) each imply the other, using the thermodynamic equations in $(2)_2$ and (3). It is a relation between η and x, y, z, t. For the ideal gas, (7) can be written explicitly as

$$\eta - \eta_0 = c_p \ln\left(\frac{1}{g}\frac{\partial \phi}{\partial z}\right), \tag{8}$$

which expresses η as a function $\eta(x, y, z, t)$.

The absolute temperature function $\tau(\eta, p)$ defined by $(2)_1$ can be used to define a so-called potential temperature $\tau(\eta, p_0) = \theta(\eta)$ (say), which is the temperature which the particle would have if it retained its current value of entropy, but replaced its pressure by the reference value p_0. Let $\theta_0 = \theta(\eta_0) = \tau(\eta_0, p_0)$ denote the reference temperature. The exponential in the ideal gas law implies $\sigma/\sigma_0 = \theta/\theta_0$, in which case (7) can also be written

$$\frac{\partial \phi}{\partial z} = \frac{g\theta}{\theta_0}. \tag{9}$$

We shall use the superposed dot to denote time differentiation following the particle in a motion (1), i.e. partial differentiation with respect to t of any function of a, b, c and t.

Attention will be confined to motions (1) which are adiabatic. The energy balance equation for such flows is

$$\dot{\eta} = 0. \tag{10}$$

This means that η is a constant for each particle, and so are σ, θ and $\partial\phi/\partial z$. These constants can vary from particle to particle, however, for example according to functions such as $\eta(a, b, c)$. When such a variation is monotonic, that parameter is available for use as an alternative particle label.

2.2 Shallow water theory

In this context the *ab initio* definition of a fluid particle which is free to move in three dimensions is transmuted, as described below, into a definition of a particle which is only free to move in two horizontal dimensions. We use the same cartesian reference frame as in §2.1, with x and y axes horizontal and fixed to the Earth, but a different letter n (for "normal") instead of h to denote the vertical coordinate. There is no thermodynamics in the previous sense, although a formal analogue of it is available.

The simplest model assigns the same constant density ρ to every particle, so that the fluid is incompressible ("water") and unstratified. The flow takes place between a bed $n = 0$ (this assumption of a flat unshifting bed could be relaxed) and a free surface $n = s(x, y, t)$, where the function on the right is unknown in advance.

Vertical momentum balance is again expressed by the hydrostatic equation $\partial p/\partial n + \rho g = 0$, in which the unknown function $p(x, y, n, t)$ is the pressure, now defined mechanically rather than thermodynamically, and g is the previous constant.

Vertical integration of the hydrostatic equation, at each fixed horizontal (x, y) station, gives $p = \rho g(s - n)$ when $p = 0$ at $n = s$ is assumed (by analogy with the definition of the pseudo-height H_0/g of the atmosphere in §2.1). A second vertical integration expresses the vertically aggregated pressure as

$$\int_0^s p \mathrm{d}n = \tfrac{1}{2}\rho g s^2.$$

This well known analogue of $p = k\rho^\gamma$ in §2.1 is the starting point for some exact cusp and swallowtail geometrical properties (Sewell and Porter, 1980; Broad, Porter and Sewell, 1994).

Shallow water theory, when regarded as a lowest order approximation in a formal perturbation expansion of the governing equations (Stoker, 1957, p.30), is

completed by the assumption that the fluid velocity is horizontal and independent of n.

Such a motion is defined by two equations

$$x = x(a, b, t), \quad y = y(a, b, t) \tag{11}$$

relating the values of x and y on the left to functions, usually unknown in advance, of a, b and t on the right. Here the labels a and b are the values of x and y at $t = 0$. They identify the transmuted particle as representing the motion of all those real particles which belong to the same material vertical line, and supposedly located at a height $s/2 = h$ (say, to be notationally consistent with §2.1 above), so that the geopotential per unit mass is $gh = \phi$ again, which can be expressed as a function $\phi(x, y, t)$. The energy balance differential equation analogous to (10) is satisfied identically in this case (Broad, Porter and Sewell, 1992, equation (39)).

3 Transformation theory

In this Section we step back from the particularities of the two contexts just introduced, and we describe some mathematical properties of transformations of variables which are valid more generally. We consider coordinate transformations, lift transformations (which include contact transformations) and Legendre transformations. The purpose of such transformations is to convert one system of partial differential equations into another, and more amenable, form. Hamilton's equations will emerge as an example here. Classic references representing the transformation theory approach include the books of Forsyth (1890, 1906) and Carathéodory (1982).

Let $x_i(a_j, t)$, for $i = 1, 2, ..., n$ and $j = 1, 2, ..., n$, denote n differentiable scalar functions of $n + 1$ scalar variables a_j and t. The equations

$$x_i = x_i(a_j, t) \tag{12}$$

can be regarded as a t-dependent smooth mapping, from a space spanned by the a_j to another space spanned by the x_i. Assume that (12) have a local inverse expressible as

$$a_i = a_i(x_j, t). \tag{13}$$

With the additional assumption

$$a_i = x_i(a_j, 0) \text{ or } x_i = a_i(x_j, 0) \tag{14}$$

we shall regard the mapping (12) or (13) as a smooth kinematically possible *motion* of material particles whose initial position (at time $t = 0$) has coordinates a_j, and whose current position (at time t) has coordinates x_i. We also use the superposed dot to denote time differentiation following the particle in such motions, i.e. partial differentiation with respect to t of any function of the a_j and t. For

example, $\dot{x}_i$ denotes velocity.

Of course we shall be most concerned with cases of planar ($n = 2$) and three dimensional ($n = 3$) motion. The easiest case is when the a_i and the x_i are both cartesian coordinates measured with respect to the same set of reference axes, as illustrated by (11) with a, b, x, y replacing a_1, a_2, x_1, x_2, and by (1) with a, b, c, x, y, h replacing $a_1, a_2, a_3, x_1, x_2, x_3$ respectively. Other replacements can be envisaged, however.

3.1 Jacobian fluxes

We choose $n = 3$ in this subsection. Let v_i, for $i = 1, 2, 3$, denote three differentiable scalar functions of the x_i, and not yet necessarily velocity components. Employing the convention of summation over repeated suffixes, we define the operator

$$D = \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i}, \tag{15}$$

which can be applied to any differentiable function of the x_i and t.

Let b_i, for $i = 1, 2, 3$, denote another set of three variables, which may be distinct from the x_i and the a_i. Let $b_i(x_j, t)$ denote three differentiable scalar functions of the four variables x_j and t, so that

$$b_i = b_i(x_j, t) \text{ with } x_i = x_i(b_j, t) \tag{16}$$

defines another t-dependent smooth mapping together with its inverse. The b_i need be neither eulerian nor lagrangian coordinates, but could be either. Denote the jacobians of (16) by

$$J = \frac{\partial(b_i)}{\partial(x_j)} \text{ with } \frac{1}{J} = \frac{\partial(x_i)}{\partial(b_j)}. \tag{17}$$

The local inversion is unique where J is finite and non-zero. If this fails at isolated places ("singularities"), there may still be a multivalued global inversion. Chynoweth and Sewell (1991, equation (B3)) showed that

$$\frac{DJ}{J} = \frac{\partial Db_i}{\partial b_i} - \frac{\partial v_i}{\partial x_i}. \tag{18}$$

On the left D is applied to the function $J(x_j, t)$ which emerges from $(16)_1$. On the right D is applied to $(16)_1$, after which $(16)_2$ is used to express the Db_i as functions of the b_j and t before the b_i-differentiation.

The v_i may or may not depend on t as well, but such dependence has not been used thus far. As stated above, the v_i may or may not have the interpretation of velocity components. Thus D may or may not signify time differentiation following a particle, although this is one possible interpretation which is common in applications.

To obtain this most familiar special case we choose $v_i = \dot{x}_i$, which implies $D\lambda = \dot{\lambda}$ for any λ by the chain rule. If we also choose $b_i = a_i$, so that $Db_i = \dot{a}_i = 0$ by definition, then (18) becomes

$$\frac{\dot{J}}{J} = -\frac{\partial \dot{x}_i}{\partial x_i}. \tag{19}$$

Different choices of the b_i in (58) and (91) yield other examples (59) and (93), respectively, of (18).

3.2 Lift transformations

Sewell and Roulstone (1994, Theorem 8) derived a family of lift transformations which encompasses ones associated with both the Legendre transformation and the geostrophic transformation of meteorology. Here we develop some consequences of particular members of that family.

To begin with, we return to the case of general n in (12), but we restrict the range of suffixes to $i = 1, 2, \ldots, m \leq n$. Later we need to distinguish between the cases $m = n$ (= 2 in shallow water theory) and $m < n$ ($m = 2$ and $n = 3$ in atmospheric flows).

Theorem 1 *Let x_i, ψ, z_i be one set of $2m + 1$ scalar variables, and let X_i, Ψ, Z_i be a second set of $2m + 1$ scalar variables, each for $i = 1, 2, \ldots, m$. Let c_{ij} be the typical component of an $m \times m$ symmetric non-singular matrix, which might depend on the x_i, ψ, z_i (and perhaps on other assignable parameters, for example t), but not separately on the X_i, Ψ, Z_i, and whose inverse has typical component c^{-1}_{ij}.*

Then the transformation

$$\left.\begin{array}{rcl} X_i &=& c^{-1}_{ij} x_j + z_i, \\ \Psi &=& \psi + \frac{1}{2} c_{ij} z_i z_j, \\ Z_i &=& c_{ij} z_j \end{array}\right\} \tag{20}$$

is a mapping from the space spanned by the first set of variables into the space spanned by the second set, with differentials which satisfy

$$\mathrm{d}\Psi - Z_i \mathrm{d}X_i = \mathrm{d}\psi - z_i \mathrm{d}x_i + \left(\frac{1}{2} z_i + c^{-1}_{ik} x_k\right) z_j \mathrm{d}c_{ij} \tag{21}$$

when all the quantities are allowed to vary.

Proof

This is a straightforward manipulation, which uses the Kronecker delta $\delta_{ij} = c_{ik} c^{-1}_{kj}$.

□

Theorem 2 *When*

$$\left(\frac{1}{2}z_i + c_{ik}^{-1}x_k\right) z_j \mathrm{d}c_{ij} = 0, \tag{22}$$

and in particular when the c_{ij} are held fixed, so that $\mathrm{d}c_{ij} = 0$,

$$\mathrm{d}\Psi - Z_i \mathrm{d}X_i = \mathrm{d}\psi - z_i \mathrm{d}x_i. \tag{23}$$

Proof

This is immediate from (21).

□

Sewell and Roulstone (1994, Theorem 8) show that (20) with (22) is not the only transformation which satisfies (23), but it is adequate for our needs here when the Coriolis parameter f is constant. It will emerge that the special choice

$$c_{ij} = \mu\delta_{ij} \tag{24}$$

with $m = 2$ and with constant scalar μ (especially $\mu = 1$) is particularly appropriate to f-plane semi-geostrophic theory, but for the present we retain the more general choice of c_{ij} to exhibit the mathematical structure of the theory.

We call (20) with (22) a *lift transformation* in the sense that (23) is satisfied. It is not necessary that both sides of (23) be zero, although that possible consequence is the motivation for the terminology.

It will turn out to be the analytical structure of the lift transformation, rather than its geometrical attributes, which is more important in guiding our theory in this paper.

3.3 Legendre transformation

For simplicity suppose, in this and the next subsection, that $m = n$. Introduce the new hypotheses that there exist relations

$$\psi = \psi(x_i, t), \quad c_{ij} = c_{ij}(t) \tag{25}$$

between the stated variables, where the functions on the right may be either known, or unknown and to be sought from further considerations. Define the new function

$$P(x_i, t) = \tfrac{1}{2}c_{ij}^{-1}x_i x_j + \psi. \tag{26}$$

A straightforward manipulation using $(20)_1$ and $(20)_2$ shows that, in value,

$$\Psi = P - x_i X_i + \tfrac{1}{2}c_{ij}X_i X_j. \tag{27}$$

Theorem 3 *Introduce the new hypothesis that*

$$z_i = \frac{\partial \psi}{\partial x_i}. \tag{28}$$

Then $(20)_1$ *can be written*

$$X_i = \frac{\partial P}{\partial x_i}. \tag{29}$$

This gradient version is the starting point for generating a Legendre transformation to a dual function $S(X_i, t)$ *with the properties*

$$x_i = \frac{\partial S}{\partial X_i}, \quad P + S = x_i X_i, \quad \frac{\partial P}{\partial t} + \frac{\partial S}{\partial t} = 0. \tag{30}$$

Then there exists a function

$$\Psi(X_i, t) = \tfrac{1}{2} c_{ij} X_i X_j - S(X_i, t) \tag{31}$$

such that $(20)_3$ *implies*

$$\frac{\partial \Psi}{\partial X_i} = Z_i. \tag{32}$$

Proof

Inserting (28) into $(20)_1$ gives (29) using (26). Applying the inverse function theorem, wherever it is locally valid, to (29) allows the x_i to be expressed as functions $x_i(X_i, t)$ at each fixed t. The chain rule shows that this has the form $(30)_1$ where $S(X_i, t) = X_i x_i(X_j, t) - P(x_i(X_j, t), t))$. Then (31) follows from (27), and (32) from (20). Releasing the passive t and using the chain rule again gives $(30)_3$.

□

Local singularities, i.e. where the inverse function theorem locally fails, can often be allowed. They can permit multivaluedness of either $P(x_i, t)$ or $S(X_i, t)$, including swallowtail forms like those in Fig. 2, as illustrated by Sewell (1987), and by Chynoweth and Sewell (1989) in the context of atmospheric fronts.

It can also be shown that, at each t, wherever $P(x_i)$ is convex, so is $S(X_i)$, and *vice versa*.

3.4 Examples of lift transformations

It is helpful at this point to give a geometrical picture of what is being achieved by marrying lift and Legendre transformations in the way established in the last Theorem.

The simplest case is when t is not explicitly present in the functions on the right of (25) and (28). Then a surface $\psi = \psi(x_i)$ lying in x_i, ψ space is lifted into

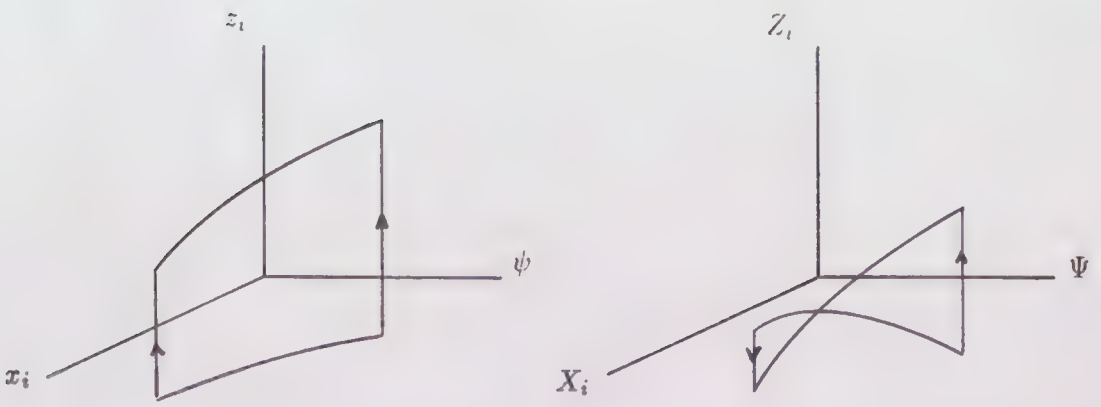

Figure 1: Example of a lift transformation

another surface in x_i, ψ, z_i space by defining $z_i = \partial\psi/\partial x_i$, as shown (for $n = 1$) in Fig. 1, and having the property $\mathrm{d}\psi - z_i \mathrm{d}x_i = 0$. (Intersection of lifts then represents contact of their projections.)

The combined effect of the lift and Legendre transformations is to produce a surface $\Psi = \Psi(X_i)$, lying in X_i, Ψ space, which is itself lifted into another surface in X_i, Ψ, Z_i space with the property $Zi = \partial\Psi/\partial X_i$, so that $\mathrm{d}\Psi - Z_i \mathrm{d}X_i = 0$. (In lift geometry there is a sense in which a hypothesis like $\psi = \psi(x_i)$ confines discussion to regular points only (see (5) of Sewell and Roulstone, 1994) where auxiliary parameters are not needed to describe the surface. Non-regular points where $\mathrm{d}\psi = z_i \mathrm{d}x_i$ still holds can be of intrinsic interest, but we shall not digress to discuss them here.)

Any kinematically possible motion (12) will generate, for each particle, a curve and a lifted curve, each parameterized by t on the respective surfaces in x_i, ψ and x_i, ψ, z_i spaces. It will map into another curve and another lifted curve on the corresponding surfaces in X_i, Ψ and X_i, Ψ, Z_i spaces.

When t *is* explicitly present on the right of (25) and (28), there is a *family* of surfaces $\psi = \psi(x_i, t)$ lying in x_i, ψ space, one for each value of t, each of which may be said to be lifted into another surface in x_i, ψ, z_i space by using $z_i = \partial\psi/\partial x_i$ as

the defining property of lift, but now having the property

$$\mathrm{d}\psi - z_i \mathrm{d}x_i = \frac{\partial \psi}{\partial t}\mathrm{d}t \tag{33}$$

when t is allowed to vary. In particular, differentiation following a motion (12) implies $\dot{\psi} - z_i \dot{x}_i = \partial\psi/\partial t$. Such a motion can still be pictured as generating space curves in both x_i, ψ and x_i, ψ, z_i space, but the second curve is no longer the lift of the first, and neither lies on a single t = *constant* surface.

The combined effect of the transformations produces a family of surfaces $\Psi = \Psi(X_i, t)$ defined by (31) and lying in X_i, Ψ space, one for each t, each of which may be said to be lifted into another surface in X_i, Ψ, Z_i by using (32) as the defining property of lift, but now having the property

$$\mathrm{d}\Psi - Z_i \mathrm{d}X_i = \frac{\partial \Psi}{\partial t}\mathrm{d}t \tag{34}$$

when t is allowed to vary. In particular, differentiation following a motion (12) implies $\dot{\Psi} - Z_i \dot{X}_i = \partial\Psi/\partial t$. Such a motion generates space curves in X_i, Ψ and X_i, Ψ, Z_i spaces, but again the second is not the lift of the first.

Theorem 4 *The partial t derivatives of (25) and (31) are related by*

$$\frac{\partial \Psi}{\partial t} = \frac{\partial \psi}{\partial t} + \left(\frac{1}{2} z_i + c_{ik}^{-1} x_k\right) z_j \dot{c}_{ij}. \tag{35}$$

Proof

This follows from Theorem 1 with (33) and (34).

Alternatively, differentiating (31) and using $(30)_3$ and (26) gives

$$\frac{\partial \Psi}{\partial t} = \frac{1}{2}\dot{c}_{ij} X_i X_j + \frac{1}{2}\dot{c}_{ij}^{-1} x_i x_j + \frac{\partial \psi}{\partial t}.$$

The result then follows from $(20)_1$ and the time derivatives of $c_{ik} c_{kj}^{-1} = \delta_{ij}$.

□

It is evident from Theorems 2 and 4 that the hypothesis of constant c_{ij} is sufficient to ensure that

$$\mathrm{d}\Psi - Z_i \mathrm{d}X_i = \mathrm{d}\psi - z_i \mathrm{d}x_i \text{ and } \frac{\partial \Psi}{\partial t} = \frac{\partial \psi}{\partial t}. \tag{36}$$

These are satisfied in the geophysical applications in view, where $c_{ij} = \delta_{ij}$ is common. Although $(36)_1$ is conventionally used in the formal definition of a lift transformation, as described after Theorem 2, it is evident that the transformation of a lift into another lift becomes diffuse when t is present explicitly in $\psi(x_i, t)$, even when it is absent from c_{ij}.

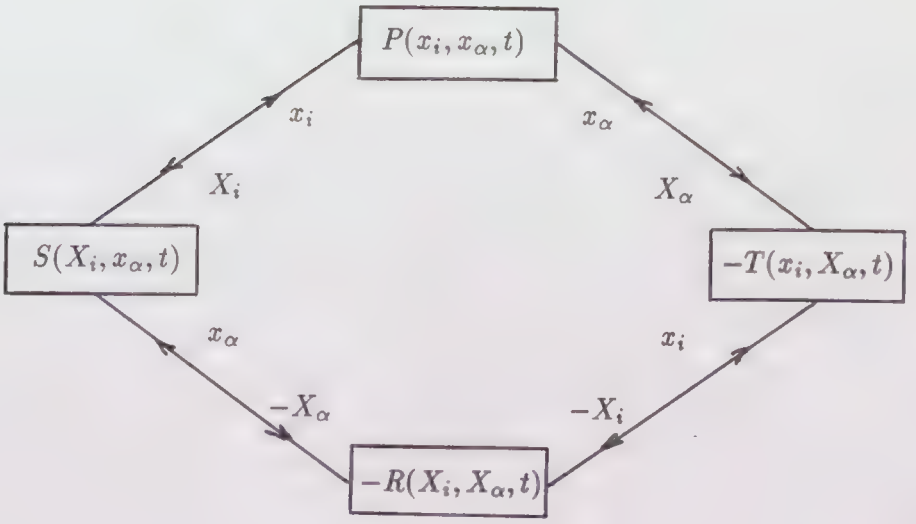

Figure 2: Quartet of Legendre transformations.

3.5 Augmented lift and Legendre transformations

Now we suppose that $m < n$, i.e. that the kinematically possible motions (12) contain $n - m$ more variables than appear in the lift transformations (20). To distinguish between the two sets of variables, we confine Latin suffixes to range from $i = 1, 2, ..., m$ and we introduce Greek suffixes to range from $\alpha = m+1, ..., n$.

In place of (25) and (28) we introduce hypotheses

$$\psi = \psi(x_i, x_\alpha, t), \quad z_i = \frac{\partial \psi}{\partial x_i}, \quad c_{ij} = c_{ij}(t) \tag{37}$$

and definitions

$$P(x_i, x_\alpha, t) = \tfrac{1}{2} c_{ij}^{-1} x_i x_j + \psi, \quad X_\alpha = \frac{\partial P}{\partial x_\alpha}. \tag{38}$$

Then (27) still applies, and Theorem 3 is augmented as follows.

Theorem 5 *This function $P(x_i, x_\alpha, t)$ is the starting point for generating the four Legendre transformations summarized schematically in Fig.2, in terms of three other functions $S(X_i, x_\alpha, t), R(X_i, X_\alpha, t)$ and $T(x_i, X_\alpha, t)$, whose values and gradients satisfy the following equations.*

$$P + S = x_i X_i, \quad X_i = \frac{\partial P}{\partial x_i}, \quad x_i = \frac{\partial S}{\partial X_i}, \tag{39}$$

$$S - R = -x_\alpha X_\alpha, \quad X_\alpha = -\frac{\partial S}{\partial x_\alpha}, \quad x_\alpha = \frac{\partial R}{\partial X_\alpha}, \tag{40}$$

$$R + T = X_i x_i, \quad x_i = \frac{\partial R}{\partial X_i}, \quad X_i = \frac{\partial T}{\partial x_i}, \tag{41}$$

$$T - P = -X_\alpha x_\alpha, \quad x_\alpha = -\frac{\partial T}{\partial X_\alpha}, \quad X_\alpha = \frac{\partial P}{\partial x_\alpha}, \tag{42}$$

$$\frac{\partial P}{\partial t} + \frac{\partial S}{\partial t} = \frac{\partial S}{\partial t} - \frac{\partial R}{\partial t} = \frac{\partial R}{\partial t} + \frac{\partial T}{\partial t} = \frac{\partial T}{\partial t} - \frac{\partial P}{\partial t} = 0. \tag{43}$$

Proof

This is a straightforward extension of the proof of Theorem 3, as explained by Sewell (1987) and Chynoweth and Sewell (1989, 1991), where it is also shown how convexity and concavity properties can be transmitted round the quartet of functions.

□

Each of $(39)_2$, $(39)_3$, $(41)_2$ and $(41)_3$ is an alternative version of $(20)_1$ with $(37)_2$, which in meteorology can be identified with the geostrophic transformation.

Fig.3 shows two simple examples of how the Legendre transformation works when the x_i and X_i are active, and the x_α and t are passive. These are adapted from Figs.2 and 3 of Chynoweth and Sewell (1989), where more general examples are also described. Suppose that t is fixed, and that we also fix all but one of the x_α, and all but one of the X_i and of the associated x_i. The function

$$S(X_i, x_\alpha) = \frac{1}{12}X_i^4 + \frac{1}{2}x_\alpha X_i^2 \tag{44}$$

of the remaining variables then has the shape shown in Fig.3(a). The long dashes indicate the locus of S-minima at each fixed x_α, and the short dashes indicate the locus of inflexions. The Legendre transformation maps (44) into the function $P(x_i, x_\alpha)$ shown in Fig.3(b). The inflexions have mapped into the edges of the swallowtail, and the minima have mapped into the self-intersection line. The convexification of (44) is the weakly convex function $S(X_i, x_\alpha)$ shown in Fig.3(c), and the Legendre transform of that is the convex function $P(x_i, x_\alpha)$ shown in Fig.3(d), in which the previous self-intersection has become a gradient discontinuity, such as may appear at an atmospheric front. The gradient jumps there represent jumps in temperature and wind velocity.

Theorem 6 *The values of the function $P(x_i, x_\alpha, t)$ are related to the values of Ψ in $(20)_2$ by (27) as before. Therefore the values of $S(X_i, x_\alpha, t)$ and $R(X_i, X_\alpha, t)$ are related to those of Ψ by*

$$\Psi = \tfrac{1}{2}c_{ij}X_iX_j - S = \tfrac{1}{2}X_iX_j - R + X_\alpha\frac{\partial R}{\partial X_\alpha}. \tag{45}$$

The first and second expressions here thereby define two different functions having the same value, which we write as $\Psi_S(X_i, x_\alpha, t)$ and $\Psi_R(X_i, X_\alpha, t)$ respectively. When $(20)_3$ is presumed as well, their gradients are

$$\frac{\partial \Psi_S}{\partial X_i} = Z_i, \quad \frac{\partial \Psi_S}{\partial x_\alpha} = X_\alpha, \tag{46}$$

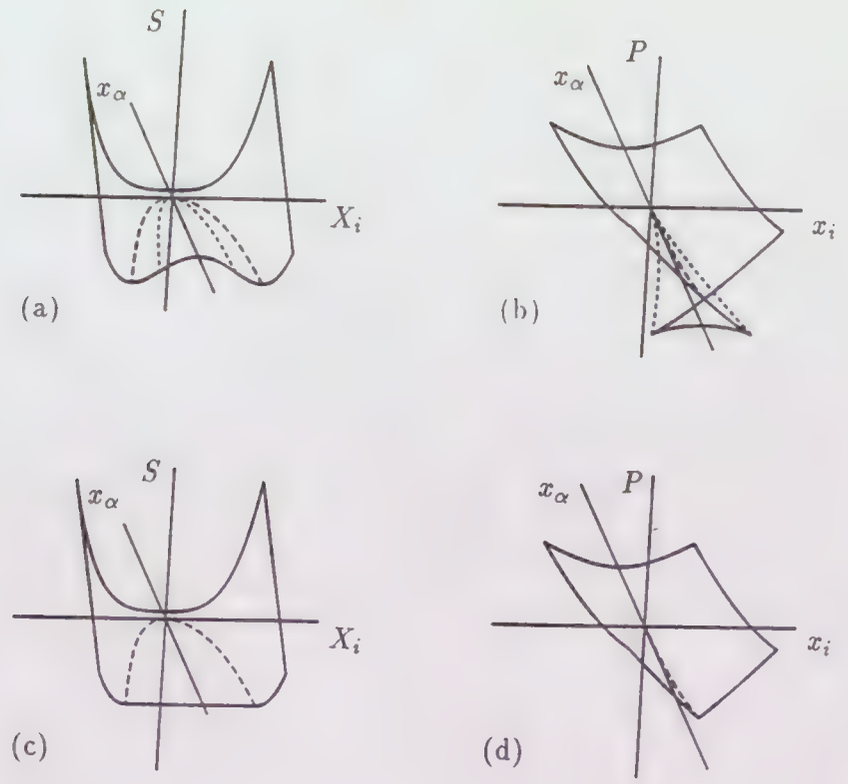

Figure 3: Examples of Legendre dual functions.

$$\frac{\partial \Psi_S}{\partial t} = \frac{\partial \psi}{\partial t} + \left(\frac{1}{2} z_i + c_{ik}^{-1} x_k\right) z_j \dot{c}_{ij}, \tag{47}$$

$$\frac{\partial \Psi_R}{\partial X_i} = Z_i + X_\beta \frac{\partial^2 R}{\partial X_\beta \partial X_i}. \quad \frac{\partial \Psi_R}{\partial X_\alpha} = X_\beta \frac{\partial^2 R}{\partial X_\beta \partial X_\alpha}, \tag{48}$$

$$\frac{\partial \Psi_R}{\partial t} = \frac{\partial \Psi_S}{\partial t} + X_\beta \frac{\partial^2 R}{\partial X_\beta \partial t}. \tag{49}$$

Proof

This follows from (21),(37) and (38), which imply

$$\begin{aligned} &\left(\frac{\partial \Psi_S}{\partial X_i} - Z_i\right) \mathrm{d}X_i + \frac{\partial \Psi_S}{\partial x_\alpha} \mathrm{d}x_\alpha + \frac{\partial \Psi_S}{\partial t} \mathrm{d}t \\ &= \frac{\partial \psi}{\partial x_\alpha} \mathrm{d}x_\alpha + \left(\frac{\partial \psi}{\partial t} + \left(\frac{1}{2} z_i + c_{ik}^{-1} x_k\right) z_j \dot{c}_{ij}\right) \mathrm{d}t \end{aligned} \tag{50}$$

which delivers (46) and (47). A similar calculation using $\mathrm{d}\Psi_R$ delivers (48) and (49).

□

We now have an augmented version of (33), namely

$$\mathrm{d}\psi - z_i \mathrm{d}x_i = X_\alpha \mathrm{d}x_\alpha + \frac{\partial \psi}{\partial t} \mathrm{d}t, \tag{51}$$

which leads to augmented versions of (34), for example

$$\mathrm{d}\Psi_S - Z_i \mathrm{d}X_i = X_\alpha \mathrm{d}x_\alpha + \frac{\partial \Psi_S}{\partial t}\mathrm{d}t. \tag{52}$$

The expressions in the last two equations are equal when the c_{ij} are constant. A third expression could be adjoined to (45) by defining a function $\Psi_T(x_i, X_\alpha, t)$ in terms of $T(x_i, X_\alpha, t)$.

4 Mass balance

The law of mass balance at places where the relevant functions have enough smoothness is a differential equation, but a different one in the two contexts of §2, finally expressing compressibility in one case and incompressibility in the other.

4.1 Shallow water theory

In a motion (11) the incompressibility of a column of water of height $s = 2h$ and geopotential $\phi = gh$ requires ϕ to be a function $\phi(a, b, t)$ with the property

$$\frac{\phi(a,b,0)}{\phi(a,b,t)} = \frac{\partial(x,y)}{\partial(a,b)} \tag{53}$$

where the jacobian on the right is that of the mapping (11). For brevity we write $\phi(a, b, 0) = \phi_0$ (say). It is not a serious restriction to suppose that ϕ_0 is an absolute constant, i.e. the same for every particle. The inverse $a = a(x, y, t), b = b(x, y, t)$ of (11) can express (53) in terms of another function

$$\phi(x, y, t) = \frac{\partial(a,b)}{\partial(x,y)}\phi_0. \tag{54}$$

Applying (19) (with $z = c$, a constant, to make $n = 3$) in this context leads to

$$\frac{\dot{\phi}}{\phi} + \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} = 0. \tag{55}$$

4.2 Atmospheric theory

The atmosphere is not incompressible, but the hydrostatic and Boussinesq approximations lead to it being regarded as such in adiabatic flow, as follows. Mass conservation expresses the current to initial density ratio as

$$\frac{\rho(a,b,c,t)}{\rho(a,b,c,0)} = \frac{\partial(a,b,c)}{\partial(x,y,h)}, \tag{56}$$

where the jacobian on the right is that of the inverse of (1). Applying (19) to this leads to the usual equation of continuity in the form

$$\frac{\dot{\rho}}{\rho} + \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{h}}{\partial h} = 0. \tag{57}$$

However, this is in terms of the real height, and we need to express it in terms of the pseudo-height z introduced in (4).

To do this, we first need an example of (18) in which $v_i = \dot{x}_i$ so that $D\lambda = \dot{\lambda}$ for any λ, but the b_i are *not* chosen to be the particle labels a_i. Instead we choose (16) to be

$$b_1 = x, \quad b_2 = y, \quad b_3 = z(x, y, h, t) \tag{58}$$

(writing $x_1 = x, x_2 = y, x_3 = h$) and denote *this* J as $\alpha = \partial z/\partial h$ so that

$$\frac{\dot{\alpha}}{\alpha} = \left(\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}\right) - \left(\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{h}}{\partial h}\right). \tag{59}$$

But $\alpha = \sigma_0/\sigma$ from (6), so that $\dot{\alpha} = -\tau\dot{\eta}\alpha/H$ using (2) and (3), and

$$\frac{\tau\dot{\eta}}{H} = \frac{\partial \dot{h}}{\partial h} - \frac{\partial \dot{z}}{\partial z}. \tag{60}$$

Then we can rewrite (57) as

$$\frac{\partial}{\partial t}\left(\frac{\rho}{\alpha}\right) + \frac{\partial}{\partial x}\left(\frac{\rho\dot{x}}{\alpha}\right) + \frac{\partial}{\partial y}\left(\frac{\rho\dot{y}}{\alpha}\right) + \frac{\partial}{\partial z}\left(\frac{\rho\dot{z}}{\alpha}\right) = 0. \tag{61}$$

However, $(2)_2$ and (3) imply $\alpha/\rho = \sigma_0 \mathrm{d}\zeta/\mathrm{d}p$, so that ρ/α depends only on z via the inverse of (4), and therefore

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -\dot{z}\frac{\mathrm{d}}{\mathrm{d}z}\left(\ln\frac{\rho}{\alpha}\right), \tag{62}$$

as shown by Hoskins and Bretherton (1972, equation 2.7, for the ideal gas) and Chynoweth and Sewell (1991, equation 42).

Hoskins and Bretherton also comment on circumstances in which the Boussinesq approximation is valid. It neglects the right side of (33), and thereby reduces the equation of continuity to

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = 0. \tag{63}$$

In adiabatic flow (10) and (60) show that

$$\frac{\partial \dot{h}}{\partial h} = \frac{\partial \dot{z}}{\partial z}. \tag{64}$$

Evidently the hydrostatic, Boussinesq and adiabatic assumptions together imply $\dot{\rho} = 0$ in (57). The resulting incompressibility approximation for the atmosphere can be expressed as a zero divergence of either vector $\dot{x}, \dot{y}, \dot{h}$ or $\dot{x}, \dot{y}, \dot{z}$ in the respective spaces.

These are finally different in structure from (55), which expresses incompressibility in shallow water theory as if ϕ were the density in a two dimensional compressible flow.

5 Dynamics

Now we have the perspective to add the laws of momentum balance to those of mass (in §4) and energy (in §2), and also to consider generalized forms (in the sense of §3) and approximations of the resulting dynamics.

5.1 Exact equations

The exact, or so-called primitive, differential equations of momentum balance in *shallow water theory* are

$$\ddot{x} + 2\frac{\partial \phi}{\partial x} - \dot{y}f = 0, \quad \ddot{y} + 2\frac{\partial \phi}{\partial y} + \dot{x}f = 0 \tag{65}$$

in terms of the unknown geopotential function $\phi(x, y, t)$ in (54). Here the Coriolis parameter f is the spin of the bed, for example the local vertical component of the Earth's rotation. Common approximations treat f as either a given constant, or as a given function of position, particularly $f(y)$ when y is northward distance.

The exact equations of horizontal momentum balance in *atmospheric theory* are

$$\ddot{x} + \frac{\partial \phi}{\partial x} - \dot{y}f = 0, \quad \ddot{y} + \frac{\partial \phi}{\partial y} + \dot{x}f = 0 \tag{66}$$

in terms of the geopotential function $gh = \phi(x, y, z, t)$ which appears in the alternative vertical momentum balance (hydrostatic) equations (7) and (9). Again here the Coriolis quantity f may be either a given constant, or a function of position, for example $f(y)$.

5.2 Semi-geostrophic approximation

The vector defined by

$$u_g = -\frac{\kappa}{f}\frac{\partial \phi}{\partial y}, \quad v_g = \frac{\kappa}{f}\frac{\partial \phi}{\partial x}, \tag{67}$$

where $\kappa = 2$ in shallow water theory and $\kappa = 1$ in atmospheric theory, is called the geostrophic velocity or geostrophic wind. At this stage it is not more than a *notional* velocity, as it is not yet demonstrated to be the time derivative of another vector.

The semi-geostrophic approximation replaces the true accelerations in (65) and (66) by the time derivatives of (67), thus giving

$$\dot{u}_g + \kappa\frac{\partial \phi}{\partial x} - \dot{y}f = 0, \quad \dot{v}_g + \kappa\frac{\partial \phi}{\partial y} + \dot{x}f = 0, \tag{68}$$

with $\kappa = 2$ or 1 for shallow water theory or atmospheric theory respectively. This can be thought of as a *substitute problem*, which replaces the exact problem specified by (65) or completed by (66).

5.3 Résumé of semi-geostrophic equations

The semi-geostrophic shallow water problem seeks to find a motion (1) whose inverse generates a geopotential (54) satisfying

$$-\overline{\frac{2}{f}\frac{\dot{\partial\phi}}{\partial y}}+2\frac{\partial\phi}{\partial x}-\dot{y}f=0,\quad \overline{\frac{2}{f}\frac{\dot{\partial\phi}}{\partial x}}+2\frac{\partial\phi}{\partial y}+\dot{x}f=0. \tag{69}$$

The semi-geostrophic atmospheric problem seeks to find a motion

$$x=x(a,b,c,t),\ \ y=y(a,b,c,t),\ \ z=z(a,b,c,t) \tag{70}$$

obtained via (1) and $z=z(p(x,y,h,t))$ via (3) and (4), whose geopotential $\phi(x,y,z,t)$ satisfies (7) with $\dot{\eta}=0$, (63) and

$$-\overline{\frac{1}{f}\frac{\dot{\partial\phi}}{\partial y}}+\frac{\partial\phi}{\partial x}-\dot{y}f=0,\quad \overline{\frac{1}{f}\frac{\dot{\partial\phi}}{\partial x}}+\frac{\partial\phi}{\partial y}+\dot{x}f=0. \tag{71}$$

Note that

$$\overline{\frac{\dot{\partial\phi}}{\partial z}}=\frac{\partial\phi}{\partial z}\overline{\frac{\tau\dot{\eta}}{H}} \tag{72}$$

is zero when $\dot{\eta}=0$.

6 Generalized semi-geostrophic equations

Here we use the notation and theory developed in §3 to construct pertinent generalizations and transformations of the semi-geostrophic equations summarized in §5.3.

6.1 Generalized hamiltonian equations

Write $n=m$ in (12), as in §3.3 and §3.4, and suppose that c_{ij} is constant, so that (36) applies. Let J_{ij} be the typical component of a constant anti-symmetric matrix (unrelated to the J in §3.1). Hypothesize the dynamical equations

$$\dot{X}_i=J_{ij}\frac{\partial\Psi}{\partial X_j} \tag{73}$$

in the X_i space, where the gradients of the function (31) are on the right. These are a generalization of Hamilton's equations in which, for example, m is not required to be even. Solutions of them have the form

$$X_i=X_i(a_j,t). \tag{74}$$

The implication is that the functions on the right are the result of inserting suitable functions (12) into (20) with (28), as specified by the following result.

Theorem 7 *The problem (73) in X_i space is equivalent to the problem*

$$\frac{\dot{\partial\psi}}{\partial x_i} - J_{ij}c_{jk}\frac{\partial\psi}{\partial x_k} + c_{ij}^{-1}\dot{x}_j = 0 \tag{75}$$

in x_i space.

Proof

Suppose that (12) is a solution of (75), and that inserting it into (20) with (28) delivers (74). From (20) and (28) we have

$$\frac{\partial\Psi}{\partial X_i} = c_{ij}\frac{\partial\psi}{\partial x_j} \text{ and } X_i = c_{ij}^{-1}x_j + \frac{\partial\psi}{\partial x_i}. \tag{76}$$

Using the first of these, and the time derivatives of the second following the particle, shows that each of (73) and (75) implies the other.

□

Theorem 8 *Solutions of (73) or (75) have the properties*

$$\frac{\partial\dot{X}_i}{\partial X_i} = 0, \quad \dot{X}_i\frac{\partial\Psi}{\partial X_i} = 0, \quad \dot{\Psi} = \dot{\psi} - \dot{x}_i\frac{\partial\psi}{\partial x_i}. \tag{77}$$

Proof

These are immediate consequences of the antisymmetry of J_{ij}, and of $(36)_1$.

□

Next we identify circumstances in which (69) is an example of (75).

6.2 Shallow water theory on an f-plane

Here we specialize to the case in which $n = m = 2$, and establish a correspondence of the general theory with the semi-geostrophic shallow water theory of (67) and (69) in the case when f is a given constant.

Suppose that $c_{ij} = \mu\delta_{ij}$ with constant μ, and that

$$J_{ij} = \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix} \tag{78}$$

with constant λ in (73).

So far we have not needed to identify coordinates and variables in the general theory with those in either of the two physical illustrations. Now we do this by introducing constant scale factors α, β, γ and m such that

$$x_1 = \alpha x, \quad x_2 = \beta y, \quad \psi = \frac{2\gamma\phi}{g} \tag{79}$$

and

$$m\dot{X}_1 = u_g, \quad m\dot{X}_2 = v_g. \tag{80}$$

The last condition requires the (previously notional) geostrophic velocity to be an actual time derivative (or at least proportional to one, like momentum if m is thought of as mass) in X_i space. This scale factor m is quite distinct from the m introduced as a number of variables in §3.2, and no confusion need arise. Likewise the α here is unrelated to that defined in §4.2.

Theorem 9 *Equations (75) (and thus (73)) are equivalent to the semi-geostrophic equations (69), and (80) are satisfied, if and only if the scale factors α, β, γ and m satisfy*

$$\alpha = \beta, \quad \lambda\mu = f, \quad \frac{\gamma\mu}{\alpha^2} = \frac{g}{f^2}, \quad \mu = m\alpha. \tag{81}$$

Proof

Substituting (79) with(78) into (75), with $n = 2$, delivers (69) if the first three relations in (81) hold (and also if $\alpha = -\beta$ and $\lambda\mu = -f$ instead).

To require (80) to hold with (69) and (73), with $\partial\Psi/\partial X_i = c_{ij}\partial\psi/\partial x_j$ from $(20)_3$, (76) and (79), it is necessary and sufficient that $\alpha = \beta$ and $m\lambda\mu\gamma = \alpha g/f$. With the second and third of (81), this delivers $\mu = m\alpha$.

□

Except for λ and μ, any pair selected from $\alpha, \gamma, \lambda, \mu, m$ can be chosen arbitrarily to satisfy (81), with f and g regarded as given, and the remaining three are then determined, apart from possible duplicity of sign. Without loss of generality, this is equivalent to choosing arbitrarily any pair selected from λ, γ, m and requiring the third parameter to satisfy

$$\lambda\gamma m^2 = \frac{g}{f}. \tag{82}$$

Then the corresponding

$$\alpha = \frac{\gamma m f^2}{g}. \tag{83}$$

For example, one might instinctively wish to choose $\alpha = 1$ and $\lambda = 1$ to achieve the simplest versions of space coordinates x_i and of Hamilton's equations (73), which would imply $m = \mu = f$ and $\gamma = g/f^2$; but other choices are possible.

In particular, if one wishes to call the X_i *geostrophic coordinates*, a justification of this is provided by (80) with the choice $m = 1$, for then the geostrophic velocity (67) with $\kappa = 2$ is the rate of change of such geostrophic coordinates. With $m = 1$, however, we can only choose $\alpha = 1$ or $\lambda = 1$, but not both. Any constant λ can be absorbed into the Hamiltonian to redefine it as $\lambda\Psi$ in place of Ψ in (73), so these particular criteria offer

$$m = \alpha = 1 = \mu, \quad \gamma = \frac{g}{f^2}, \quad \lambda = f \tag{84}$$

as an inviting choice of parameters, with $\psi = 2\phi/f^2$.

6.3 Generalized augmented hamiltonian equations

Here we consider the case $m < n$ first allowed for explicitly in §3.5. The case $m = 2$ with $n = 3$ is required to describe the semi-geostrophic approximation of the atmospheric problem summarized in (70) to (72). Latin suffixes here range only up to m, and Greek suffixes range from $m+1$ to n.

Now let c_{ij} and J_{ij} be constant $m \times m$ matrices, with J_{ij} anti-symmetric. Use the X_i defined in (20) and the X_α defined in (38) to hypothesize the dynamical equations

$$\dot{X}_i = J_{ij}\frac{\partial \Psi_S}{\partial X_j}, \quad \dot{X}_\alpha = 0 \tag{85}$$

in terms of the function $\Psi_S(X_i, x_\alpha, t)$ defined in Theorem 6, having gradients (46). This a generalized augmented hamiltonian problem, in which again it is not essential that m be even.

Bearing in mind that $x_\alpha = \partial R/\partial X_\alpha$ from (35), the solutions of (85) for each material particle (i.e. for each set of constant values $a_1, ..., a_m, a_{m+1}, ..., a_n$ of the labels in (12)) have the form

$$X_i = X_i(t), \quad X_\alpha = \textit{constant}. \tag{86}$$

The generalized version of incompressibility (63) with the Boussinesq approximation can be stated as

$$\frac{\partial \dot{x}_i}{\partial x_i} + \frac{\partial \dot{x}_\alpha}{\partial x_\alpha} = 0. \tag{87}$$

This acts as a constraint to be imposed on the solutions (86), with the aid of the expressions in Theorems 5 and 6.

The problem (85) in X_i, X_α space is equivalent to the problem (75) with $\dot{\overline{\partial P/\partial x_\alpha}} = 0$ in x_i, x_α space. To show this requires the same manipulations as in the proof of Theorem 7, together with the additional properties in (37) and (38). Solutions of (85) have the properties (77) but with Ψ there replaced by Ψ_S.

6.4 Atmospheric theory on an f-plane

Let $m = 2$ and $n = 3$ in the previous subsection. Let $c_{ij} = \mu\delta_{ij}$, and J_{ij} be as in (78), with constant μ and λ. Introduce constant scale factors α, β and γ such that the variables x_1, x_2 and ψ in the general theory begun in Theorem 1 are related by

$$x_1 = \alpha x, \quad x_2 = \beta y, \quad \psi = \frac{\gamma\phi}{g} \tag{88}$$

to the horizontal coordinates x and y and the geopotential ϕ in (70) and (71).

The variables X_1 and X_2 are required to satisfy $(20)_1$ with $(37)_2$. Introduce the choices

$$x_3 = \nu h, \;\; X_3 = \eta, \;\; \nu = \frac{\gamma}{\eta}. \tag{89}$$

Here ν is a constant scale factor which relates the real height h to the only surviving member of the set of x_α variables introduced in §3.5. We shall seek to model only adiabatic flows, for which the choices in (89) are consistent with $(85)_2$. (We could have chosen X_3 to be an arbitrary function of η, but that yields no extra advantage.)

Theorem 10 *Equations (85) are equivalent to the semi-geostrophic equations (71) and the adiabatic condition, together with the incompressibility condition*

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{h}}{\partial h} = 0, \tag{90}$$

and (80) are satisfied, if and only if the scale factors satisfy (81), provided the choice of ν satisfies (89).

Proof

The proof of Theorem 9 applies as it stands to identify the X_1 and X_2 hamiltonian equations in (85) with (71). The scale factors α, α, ν associated with the three cartesian physical space coordinates appear in both numerator and denominator of (87) and therefore (90), and so cancel out. The adiabatic condition $\dot{\eta} = 0$ is equivalent to $\dot{X}_3 = 0$. From (38) and $(88)_3$ we have

$$X_3 = \frac{\partial \psi}{\partial x_3} = \frac{\gamma}{g} \frac{\partial \phi}{\partial x_3}.$$

For adiabatic flow, this is consistent with the choices (89) because $\phi = gh$.

□

6.5 Potential vorticity and the Monge-Ampère equation

The geostrophic momentum transformation, in the generalized notation, is a mapping

$$X_i = X_i(x_j, x_\beta, t), \;\; X_\alpha = X_\alpha(x_j, x_\beta, t) \tag{91}$$

at each t from x_j, x_β space to X_i, X_α space. It is defined by equations (20) with(28) if $m = n$, or by $(39)_2$ and $(38)_2$ if $m < n$. Let q denote the $n \times n$ Jacobian of this mapping. Clearly q is also the Hessian, at each t, of the function $P(x_i, t)$ in (26) or $P(x_i, x_\alpha, t)$ in $(38)_1$, respectively.

When $n = 3$ and $m = 2$, we can regard (91) as a particular example of $(16)_1$, in which case (18) becomes

$$\frac{Dq}{q} = \frac{\partial DX_i}{\partial X_i} + \frac{\partial DX_\alpha}{\partial X_\alpha} - \left(\frac{\partial v_i}{\partial x_i} + \frac{\partial v_\alpha}{\partial x_\alpha} \right). \tag{92}$$

If we also choose $v_i = \dot{x}_i, v_\alpha = \dot{x}_\alpha$ to be particle velocity, (92) becomes

$$\frac{\dot{q}}{q} = \frac{\partial \dot{X}_i}{\partial X_i} + \frac{\partial \dot{X}_\alpha}{\partial X_\alpha} - \left(\frac{\partial \dot{x}_i}{\partial x_i} + \frac{\partial \dot{x}_\alpha}{\partial x_\alpha} \right) \tag{93}$$

as in (20) of Chynoweth and Sewell (1991).

Theorem 11 *When (85) and (87) apply for $n = 3$ and $m = 2$, $\dot{q} = 0$.*

Proof

This is immediate from (93).

□

This result proves the conservation of q following the particle. In the atmospheric case q is identifiable with the potential vorticity (see Hoskins, 1975, §3(ii) and (iii)). The structure of the conservation proof is clearly displayed *via* (93). Liouville's theorem makes $\partial \dot{X}_i / \partial X_i = 0$; the adiabatic assumption makes $\partial \dot{X}_\alpha / \partial X_\alpha = 0$; the Boussinesq approximation and incompressibility (63) justify (87).

At each t, the Jacobian of the inverse of (91) is q^{-1}. Because $P(x_i, x_\alpha, t)$ and $R(X_i, X_\alpha, t)$ are Legendre dual functions, by (39) and (40), q^{-1} is also the Hessian of $R(X_i, X_\alpha, t)$, i.e.

$$\det \left(\frac{\partial^2 R}{\partial (X_i, X_\alpha)} \right) = q^{-1}. \tag{94}$$

In developing solution strategies for the semi-geostrophic equations in the atmosphere, Cullen *et al.* (*op. cit.*) use the conservation of q as the starting point for an assumption that q can be regarded as an assigned function $q(X_i, X_\alpha)$ at each t, in which case (94) becomes a Monge-Ampère equation for the determination of $R(X_i, X_\alpha, t)$ regarded as an unknown.

When $n = m = 2$, instead of (93) we obtain

$$\frac{\dot{q}}{q} = \frac{\partial \dot{X}_i}{\partial X_i} - \frac{\partial \dot{x}_i}{\partial x_i}. \tag{95}$$

If the incompressibility assumption (87) is replaced by a compressibility assumption

$$\frac{\dot{\phi}}{\phi} + \frac{\partial \dot{x}_i}{\partial x_i} = 0 \tag{96}$$

consistent with (55), a quantity different from q is conserved.

Theorem 12 *When $(85)_1$ and (96) apply for $n = m = 2$, q/ϕ is conserved following a particle.*

Proof
Liouville's theorem and (96) imply $\dot{\overline{q/\phi}} = 0$ from (95).

□

This result would be one reason for calling q/ϕ, instead of q, the potential vorticity in this context. A completely different proof is offered by Salmon (1985, equation (3.28)), building upon Salmon (1983, §2).

7 Variable Coriolis parameter

The problem of identifying what may be regarded as essential elements in the mathematical structure of semi-geostrophic equations for cases in which the Coriolis parameter is not a fixed constant is broached here. We shall not offer a complete solution, but we shall use the rather clear cut structure exposed above for a constant Coriolis parameter to develop certain pointers.

7.1 Planetary semi-geostrophic equations

Here we explore how the form of some equations obtained by Shutts (1989) for motion on the surface of a sphere can be recovered, by quite a different method to his.

We begin by hypothesizing Hamilton's equations in the abstract form (73), but in the particular case $n = m = 2$ with (78), namely

$$\dot{X}_1 = -\lambda \frac{\partial \Psi}{\partial X_2}, \quad \dot{X}_2 = \lambda \frac{\partial \Psi}{\partial X_1}. \tag{97}$$

Here Ψ is a given function $\Psi(X_1, X_2, t)$, and λ is a constant.

Next we suppose that there is a pair of abstract variables x_1 and x_2, and another function $\psi(x_1, x_2, t)$, these being related to the foregoing variables by the particular lift transformation

$$\left.\begin{aligned} X_1 = x_1 + \frac{\partial \psi}{\partial x_1}, \quad X_2 = x_2 + \frac{\partial \psi}{\partial x_2}, \\ \Psi = \psi + \frac{1}{2}\left(\left(\frac{\partial \psi}{\partial x_1}\right)^2 + \left(\frac{\partial \psi}{\partial x_2}\right)^2\right), \\ \frac{\partial \Psi}{\partial X_1} = \frac{\partial \psi}{\partial x_1}, \quad \frac{\partial \Psi}{\partial X_2} = \frac{\partial \psi}{\partial x_2}. \end{aligned}\right\} \tag{98}$$

This corresponds to the case $\mu = 1$ in (24) and (20), with (76).

The differential equations (97) become

$$\dot{x}_1 + \dot{\overline{\frac{\partial \psi}{\partial x_1}}} = -\lambda \frac{\partial \psi}{\partial x_2}, \quad \dot{x}_2 + \dot{\overline{\frac{\partial \psi}{\partial x_2}}} = \lambda \frac{\partial \psi}{\partial x_1} \tag{99}$$

in the x_1, x_2 space. The Legendre transformation in Theorem 3 tells us that

$$\left.\begin{aligned} \Psi(X_1, X_2, t) &= \frac{1}{2}({X_1}^2 + {X_2}^2) - S(X_1, X_2, t), \\ \psi(x_1, x_2, t) &= P(x_1, x_2, t) - \frac{1}{2}({x_1}^2 + {x_2}^2) \end{aligned}\right\} \tag{100}$$

in terms of the Legendre dual functions $S(X_1, X_2, t)$ and $P(x_1, x_2, t)$.

Now we hypothesize relations between such abstract variables, and the cartesian coordinates x and y representing local eastward and northward distances in the tangent plane at the Earth's surface, with a geopotential function $\phi(x, y, t)$. We suppose that the Coriolis parameter f is a given function $f(y)$, and that a particular value f_0 (say) of $f(y)$ is present as well.The geostrophic velocity is now defined by (67) with $\kappa = 1$ but with the *variable* $f(y)$ in each denominator.

The required relations are set down as scaling relations like those in (79) or (88) with (80), but with the difference that some of the scale factors are now allowed to be certain functions of y at least, instead of the constants which they were when f was constant.

Firstly, however, we choose some of the scale factors to be constants again, namely

$$\lambda = \alpha = f_0, \text{ so that } x_1 = f_0 x, \tag{101}$$

together with

$$\dot{X}_1 = f_0 u_g. \tag{102}$$

Instead of $(79)_2$ and $(80)_2$, however, we choose

$$x_2 = \int_{y_0}^{y} f(u)\mathrm{d}u, \text{ so that } \frac{\mathrm{d}x_2}{\mathrm{d}y} = f(y) \tag{103}$$

and

$$\dot{X}_2 = f(y) v_g. \tag{104}$$

Finally, in place of $(79)_3$ or $(88)_3$ we choose

$$\psi(x_1, x_2) = \phi(x, y). \tag{105}$$

Theorem 13 *The partly y-dependent scalings set down in (101) to (105) convert Hamilton's equations (97) into*

$$\dot{u}_g + \frac{\partial \phi}{\partial x} - f(y)\dot{y} = 0, \quad \frac{f(y)}{f_0}\overline{\left(\frac{\dot{f(y)}v_g}{f_0}\right)} + \frac{\partial \phi}{\partial y} + f(y)\dot{x} = 0, \tag{106}$$

which are representations of zonal and meridional planetary semi-geostrophic equations.

Proof

We have already shown that (97) are equivalent to (99) by using the lift transformation (98). The transformations (101) and (103) show that the choice (105) has the properties

$$\frac{\partial\phi}{\partial x} = f_0\frac{\partial\psi}{\partial x_1}, \quad \frac{\partial\phi}{\partial y} = f(y)\frac{\partial\psi}{\partial x_2}. \tag{107}$$

Then (99) becomes

$$f_0\dot{x} + \frac{1}{f_0}\frac{\dot{\partial\phi}}{\partial x} + \frac{f_0}{f(y)}\frac{\partial\phi}{\partial y} = 0, \quad f(y)\dot{y} + \frac{\dot{1}}{f(y)}\frac{\partial\phi}{\partial y} - \frac{\partial\phi}{\partial x} = 0. \tag{108}$$

In reverse order these are equivalent to (106), using the definitions

$$u_g = -\frac{1}{f(y)}\frac{\partial\phi}{\partial y}, \quad v_g = \frac{1}{f(y)}\frac{\partial\phi}{\partial x}. \tag{109}$$

Suppose now that φ and Ω denote latitude and the spin of the Earth, respectively. If, in (108), we replace $f(y)$ by $2\Omega\sin\varphi$, and f_0 by 2Ω, we obtain the meridional and zonal equations of Shutts (*op. cit.*, equations (77) and (78)).

□

7.2 Pseudo-plane approximations

It is clear that the planetary semi-geostrophic equations (106) are *not* obtained as in §5.2, i.e. not just by replacing the acceleration in the exact equations by the time derivatives of the geostrophic velocity. This is not surprising.

There is an intermediate situation, between the planetary case and the case of constant f. This is when variable f is approximated as

$$f(y) = \beta y \tag{110}$$

with constant β (quite distinct from the β used in (79), (81) and (88)). This is called the β-plane approximation, with northward y.

Again we might begin with two sets of abstract variables X_1, X_2, Ψ and x_1, x_2, ψ related by the lift transformation (98). Suppose that pseudo-Hamiltonian equations

$$\dot{X}_1 = -f(y)\frac{\partial\Psi}{\partial X_2}, \quad \dot{X}_2 = f(y)\frac{\partial\Psi}{\partial X_1} \tag{111}$$

are postulated. Here the variable $f(y)$ takes the place of the constant scale factor λ in (97), so that Liouville's theorem will no longer apply in general. Even so, we can ask whether recognizable semi-geostrophic equations can be recovered from (99), which holds with $\lambda = f(y)$, by postulating nonlinear relations between the abstract x_1, x_2 and the cartesian position coordinates x, y.

If we repeat the choice (103) with the particular $f(y) = \beta y$,

$$x_2 = \tfrac{1}{2}\beta(y^2 - y_0^2) \text{ and } f = \left(2\beta x_2 + \beta^2 y_0^2\right)^{\frac{1}{2}}. \tag{112}$$

We use this to define an intermediate function $\Phi(x_1, y)$, say, by $\Phi(x_1, y) = \psi(x_1, x_2)$, in terms of which (111) implies

$$-\overline{\frac{1}{f(y)}\frac{\partial\Phi}{\partial y}}^{\cdot} + f(y)\frac{\partial\Phi}{\partial x_1} - \dot{y}f(y) = 0, \quad \overline{\frac{\partial\Phi}{\partial x_1}}^{\cdot} + \frac{\partial\Phi}{\partial y} + \dot{x}_1 = 0 \tag{113}$$

from (99) in reverse order. These are formally close to the semi-geostrophic equations (68). Whether they are equivalent to them, however, requires investigation of whether we can choose a mapping

$$x_1 = x_1(x, y) \text{ such that } \phi(x, y) = \Phi(x_1, y), \tag{114}$$

whose properties

$$\frac{\partial\phi}{\partial x} = \frac{\partial\Phi}{\partial x_1}\frac{\partial x_1}{\partial x}, \quad \frac{\partial\phi}{\partial y} = \frac{\partial\Phi}{\partial x_1}\frac{\partial x_1}{\partial y} + \frac{\partial\Phi}{\partial y} \tag{115}$$

lead to the desired result.

The difficulty which emerges is that, while the properties

$$\frac{\partial x_1}{\partial x} = f(y), \quad \frac{\partial x_1}{\partial y} = 0 \tag{116}$$

would be sufficient to allow (113) to be written as (68) with $\kappa = 1$ and $f = f(y)$, as may be easily verified, there is no mapping $(114)_1$ which could have the properties (116), for example because (116) would imply

$$\frac{\partial^2 x_1}{\partial y \partial x} \neq \frac{\partial^2 x_1}{\partial x \partial y}. \tag{117}$$

In other words, (116) are not mutually consistent, because if x_1 depends only on x, then $\partial x_1/\partial x$ would not depend on y.

One might suppose that associated difficulties can be circumvented by using so-called non-holonomic systems of equations, which in the present case would take the differential form

$$\mathrm{d}x_1 = f(y)\mathrm{d}x, \tag{118}$$

but such suggestions have not so far been explicit enough to carry conviction.

7.3 Other lift transformations

The interest which can attach to problems in which the Coriolis parameter is not a constant prompts enquiry of whether lift transformations having a similar but different structure to those in Theorem 1 can be devised, such that the new structure is useful in geophysical fluid dynamics. There are, after all, many different types of lift transformation. Sewell and Roulstone (1994) proved how those in (20) are particular members of a class deducible from a simple algebraic starting point, namely by requiring the X_i and Ψ to be quadratic functions of the x_i, ψ and z_i. Here we explore briefly a different starting point.

Again let x_i,ψ,z_i and X_i,Ψ,Z_i be two sets of $2m+1$ scalar variables. Let $u(x_i, z_i)$ be a scalar function of the $2m$ x_i and z_i, to be found from criteria to be formulated below. The transformation

$$X_i = x_i + \frac{z_i}{u}, \quad \Psi = \psi + \frac{z_i z_i}{2u}, \quad Z_i = Z_i(x_j, \psi, z_j) \tag{119}$$

is a mapping from x_i, ψ, z_i space to X_i, Ψ, Z_i space, for any $m+1$ scalar functions $u(x_i, z_i)$ and $Z_i(x_j, \psi, z_j)$ still to be specified. If u is not a constant, the functions $X_i(x_j, z_j)$ and $\Psi(x_i, \psi, z_i)$ will not be quadratic in general.

Theorem 14 *The mapping (119) is a lift transformation from x_i, ψ, z_i space to X_i, Ψ, Z_i space, in the sense that differentials satisfy*

$$\mathrm{d}\Psi - Z_i \mathrm{d}X_i = \mathrm{d}\psi - z_i \mathrm{d}x_i \tag{120}$$

for all $\mathrm{d}x_i$ and $\mathrm{d}z_i$, if and only if the functions $u(x_i, z_i)$ and $Z_i(x_j, z_j)$ satisfy

$$u\frac{\partial u}{\partial z_j} - \frac{\partial u}{\partial x_j} = 0 \tag{121}$$

and

$$\left(\delta_{ij} - \frac{z_j}{u}\frac{\partial u}{\partial z_i}\right) Z_j = \left(\delta_{ij} - \frac{z_j}{2u}\frac{\partial u}{\partial z_i}\right) z_j. \tag{122}$$

Proof

Direct calculation shows that differentials satisfy

$$\begin{aligned}
\mathrm{d}\Psi - Z_i \mathrm{d}X_i &= \mathrm{d}\psi - z_i \mathrm{d}x_i + (z_i - Z_i)\mathrm{d}x_i + \frac{(z_i - Z_i)}{u}\mathrm{d}z_i + \frac{z_i}{u^2}\left(Z_i - \frac{1}{2}z_i\right)\mathrm{d}u \\
&= \mathrm{d}\psi - z_i \mathrm{d}x_i + \left(z_i - Z_i + \frac{z_j}{u^2}\left(Z_j - \frac{1}{2}z_j\right)\frac{\partial u}{\partial x_i}\right)\mathrm{d}x_i \\
&\quad + \left(z_i - Z_i + \frac{z_j}{u}\left(Z_j - \frac{1}{2}z_j\right)\frac{\partial u}{\partial z_i}\right)\frac{\mathrm{d}z_i}{u}.
\end{aligned}$$

Hence (120) holds for any $\mathrm{d}x_i$ and $\mathrm{d}z_i$ if

$$Z_i - z_i = \frac{z_j}{u^2}\left(Z_j - \frac{1}{2}z_j\right)\frac{\partial u}{\partial x_i} = \frac{z_j}{u}\left(Z_j - \frac{1}{2}z_j\right)\frac{\partial u}{\partial z_i}.$$

Evidently $Z_j = \frac{1}{2}z_j$ would not be self-consistent, and assuming $z_j \neq 0$, (121) and (122) then follow.

□

The problem is thereby reduced to solving the m partial differential equations (121) for $u(x_i, z_i)$, inserting these solutions into (122), and solving those linear algebraic equations to find functions $Z_i(x_j, z_j)$. There will be many solutions of the system (121), and we illustrate two approaches.

An elementary approach is to look for solutions $u = v(x_i)w(z_j)$ in which the variables are separable, where $v(x_i)$ and $w(z_j)$ are two scalar functions, each of the indicated m variables, to be found.

Theorem 15 *Separable solutions of (121) must have the form*

$$u = \frac{a + c_j z_j}{b - c_i x_i} \tag{123}$$

where a, b and the c_i are $m + 2$ arbitrary constants. Then (122) have the solution

$$Z_j = z_j + \frac{c_j z_i z_i}{2a} \tag{124}$$

provided $0 \neq a \neq -c_j z_j$.

Proof
The separable assumption inserted into (121) implies $w = 0$ or

$$\frac{\partial w}{\partial z_j} = -\frac{\partial v^{-1}}{\partial x_j} = c_j$$

which gives (123). The matrix of coefficients on the left of (122) has determinant $a/(a + c_j z_j)$, whence the result (124), which can easily be verified directly.

□

The corresponding lift transformation can be written down by inserting (123) and (124) into (119).

More general solutions of (121) can be found by exploiting the idea of characteristics (e.g. in the case of $m = 1$ at first). The following result emerges. Write $s_i = z_i + u x_i$ and let $F(s_i)$ denote an arbitrary differentiable function of m variables.

Theorem 16 *Any function $u(x_i, z_i)$ which satisfies*

$$u = F(z_i + u x_i) \tag{125}$$

and

$$x_i \frac{\partial F}{\partial s_i} \neq 1 \tag{126}$$

will satisfy (121).

Proof
The partial derivatives of (125) combine to give

$$\left(u \frac{\partial u}{\partial z_j} - \frac{\partial u}{\partial x_j}\right)\left(1 - x_i \frac{\partial F}{\partial s_i}\right) = 0. \tag{127}$$

□

The inverse function theorem shows that (126) is sufficient for (125) to have at least a local solution $u(x_i, z_i)$. The corresponding $Z_i(x_j, z_j)$ are then to be found from (122).

It is clear that (123) is recoverable from (125) in the particular case of linear

$$F(s_i) = \frac{a + c_i s_i}{b} \tag{128}$$

when $0 \neq b \neq c_i x_i$.

The meteorological problem associated with these results is the following. Can physical spatial variables be linked to the x_i, for example by a linear scaling in the case $m = 2$, in such a way that the arbitrariness of the function $F(s_i)$ can be exploited so that the solution $u(x_i, z_i)$ of (125) fits a pre-assigned function $u(x_i)$ which represents, at least approximately, how the Coriolis parameter depends on position?

Put otherwise, suppose that u is an assigned function $u(f)$ of the Coriolis parameter f, and that f is an assigned function $f(x_i)$ of the x_i, so that $u(x_i) = u[f(x_i)]$ is also assigned. Suppose also that functions $z_i(x_j)$ can be hypothesized (representing the role served by (28) in the f-plane case). Then, if we refrain from assigning the function $F(s_i)$, equation (125) becomes an equation

$$u(x_i) = F[x_i u(x_i) + z_i(x_j)] \tag{129}$$

to be satisfied by $F(s_i)$ for all x_i. Under what circumstances can this be solved to find $F(s_i)$? Does (119) offer a helpful guide to a generalization of Theorem 3 to the non-f-plane case?

Dynamical systems associated with other generalizations of the geostrophic coordinate transformation have been considered in the literature, for example by Salmon (1985), Magnusdottir and Schubert (1990, 1991) and Purser (1988, 1993). Future work will be needed to study them in the context of the guidance offered by the lift structure explained here and in Sewell and Roulstone (1994).

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